1 Introduction

Biarc curve fitting determines two circular arcs passing through two given points and tangents at those points. When applied to a series of points, it determines a piecewise circular arc interpolation of given points.

This document summarizes the work of [ROSSIGNAC, REQUICHA]

2 Formulation of Jaroslaw & Aristides

Following figure depicts a biarc curve fitting through points $P_1, P_2$ with given tangents $T_1, T_2$.

apex points of two segments are given by
\[ B_1 = P_1 + a_1 T_1 \]
\[ B_2 = P_2 - a_2 T_2 \]

relations that those points must satisfy are

\[ |B_1 - P_1| = |B_{12} - B_1| = a_1 \quad (1) \]
\[ |B_2 - P_2| = |B_{12} - B_2| = a_2 \quad (2) \]
\[ |B_2 - B_1| = a_1 + a_2 \quad (3) \]

where junction points of two arcs is

\[ B_{12} = B_1 + \frac{a_1}{a_1 + a_2} (B_2 - B_1) = \frac{a_2 B_1 + a_1 B_2}{a_1 + a_2} \]

Substituting definitions of \( B_1, B_2 \) in basic equation

\[ B_2 - B_1 = (P_2 - a_2 T_2) - (P_1 + a_1 T_1) \]
\[ = P_2 - P_1 - (a_1 T_1 + a_2 T_2) \]

and defining

\[ P_2 - P_1 \equiv S \]

basic equation becomes

\[ |B_2 - B_1|^2 = (a_1 + a_2)^2 \]
\[ (P_2 - P_1 - (a_1 T_1 + a_2 T_2))^2 = (a_1 + a_2)^2 \]
\[ (S - (a_1 T_1 + a_2 T_2))^2 = (a_1 + a_2)^2 \]
\[ S^2 - 2S \cdot (a_1 T_1 + a_2 T_2) + (a_1 T_1 + a_2 T_2)^2 = (a_1 + a_2)^2 \]
\[ S^2 - 2S \cdot a_1 T_1 - 2S \cdot a_2 T_2 + a_1^2 + 2a_1a_2 T_1 \cdot T_2 + a_2^2 = a_1^2 + 2a_1a_2 + a_2^2 \]
\[ S^2 - 2S \cdot a_1 T_1 - 2S \cdot a_2 T_2 + 2a_1a_2 T_1 \cdot T_2 + a_1^2 + 2a_1a_2 + a_2^2 = a_1^2 + 2a_1a_2 + a_2^2 \]
\[ a_1 a_2 (T_1 \cdot T_2 - 1) + \frac{S^2}{2} = a_1 S \cdot T_1 + a_2 S \cdot T_2 \]

General formula is given (in [ROSSIGNAC,REQUICHA, p. 300])

\[ a_2 = \frac{a_1 (S \cdot T_1) - \frac{1}{2} \|S\|^2}{a_1 (T_1 \cdot T_2 - 1) - S \cdot T_2} \]

### 2.0.1 Specified ratio of \( a_i \)

If the ratio of two side lengths is specified

\[ \rho = \frac{a_2}{a_1} \]

then

\[ a_1 \rho a_1 (T_1 \cdot T_2 - 1) + \frac{S^2}{2} = a_1 S \cdot T_1 + \rho a_1 S \cdot T_2 \]
\[ a_1^2 [\rho (T_1 \cdot T_2 - 1)] - a_1 (S \cdot T_1 + \rho S \cdot T_2) + \frac{S^2}{2} = 0 \]

this general relation has some special cases that must be handled. These cases are described in [ROSSIGNAC,REQUICHA]
2.1 Other Relations Between Auxiliary Points

When junction point expressed in initial variables

\[ B_{12} = \frac{a_2 B_1 + a_1 B_2}{a_1 + a_2} \]

\[ = \frac{a_2 (P_1 + a_1 T_1) + a_1 (P_2 - a_2 T_2)}{a_1 + a_2} \]

\[ = \frac{a_2 P_1 + a_2 a_1 T_1 + a_1 P_2 - a_1 a_2 T_2}{a_1 + a_2} \]

\[ = \frac{a_2 P_1 + a_1 P_2 + a_2 a_1 (T_1 - T_2)}{a_1 + a_2} \]

Chords of arcs become

\[ B_{12} - P_1 = \frac{a_2 P_1 + a_1 P_2 + a_2 a_1 (T_1 - T_2)}{a_1 + a_2} - P_1 \]

\[ = \frac{a_2 P_1 + a_1 P_2 + a_2 a_1 (T_1 - T_2) - (a_1 + a_2) P_1}{a_1 + a_2} \]

\[ = \frac{a_1 (P_2 - P_1) + a_2 a_1 (T_1 - T_2)}{a_1 + a_2} \]

\[ B_{12} - P_2 = \frac{a_2 P_1 + a_1 P_2 + a_2 a_1 (T_1 - T_2)}{a_1 + a_2} - P_2 \]

\[ = \frac{a_2 P_1 + a_1 P_2 + a_2 a_1 (T_1 - T_2) - (a_1 + a_2) P_2}{a_1 + a_2} \]

\[ = \frac{a_2 (P_1 - P_2) + a_2 a_1 (T_1 - T_2)}{a_1 + a_2} \]

2.2 Computing arc center and angle

This section describes the computation of center and arc-angle of two arcs from given two points on circle \((A, C)\) and apex point \(B\). Consider following figure
If points $A$, $C$ are on circle and tangents to circle at $A$, $C$ intersect at apex $B$, then

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \cos(2\theta) = D$$
$$\overrightarrow{AB} \times \overrightarrow{BC} = \sin(2\theta) \, \mathbf{a}$$

(where $\mathbf{a}$ is arc-plane normal).

From

$$\cos(2\theta) = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$$

$$\cos \theta = \sqrt{\frac{1 + D}{2}}$$
$$\sin \theta = \sqrt{\frac{1 - D}{2}}$$
$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \sqrt{\frac{1 - D}{1 + D}}$$

from

$$\tan \theta = \frac{\overrightarrow{AB}}{r}$$

radius is determined as

$$r = \frac{\overrightarrow{AB}}{\tan \theta} = \sqrt{\frac{1 + D}{1 - D}} \overrightarrow{AB}$$

from

$$\cos \theta = \frac{r}{BO}$$

circle center-to-apex distance $BO$ is computed as

$$BO = \frac{r}{\cos \theta}$$

then center of arc is

$$O = B + \frac{r}{\cos \theta} \frac{\overrightarrow{BA} + \overrightarrow{BC}}{\| \overrightarrow{BA} + \overrightarrow{BC} \|}$$

### 2.2.1 Using complex numbers

Center can be found from the intersection of line segments through point $a$ and $c$ with directions $i(b - a), i(c - b)$ that is

$$a + \lambda i(b - a) = c + \mu i(c - b)$$

Placing $P_1, P_2$ on real axis with midpoint $\frac{P_1 + P_2}{2}$ in origin, start and points become

$$p_1 = -\frac{s}{2} + 0i$$
$$p_2 = +\frac{s}{2} + 0i$$

junction points is

$$b_{12} = \frac{a_2 \frac{s}{2} + a_1 \frac{s}{2} + a_2 a_1 \left(e^{i\theta_1} - e^{i\theta_2}\right)}{a_1 + a_2}$$
and apex points are

\[ b_1 = -\frac{s}{2} + a_1 e^{i\theta_1} \]
\[ b_2 = +\frac{s}{2} - a_2 e^{i\theta_2} \]

For the first control triangle \( \left(-\frac{s}{2}, -\frac{s}{2} + a_1 e^{i\theta_1}, \frac{a_2 - \frac{s}{2} + a_1 e^{i\theta_1}}{a_1 + a_2} (e^{i\theta_1} - e^{i\theta_2})\right) \) intersection is described by

\[-\frac{s}{2} + \lambda ie^{i\theta_1} = +\frac{s}{2} + \mu ie^{i\theta_2} \]
\[\lambda ie^{i\theta_1} - \mu ie^{i\theta_2} = s \]
\[\lambda e^{i\theta_1} - \mu e^{i\theta_2} = -is \]

\[\lambda \cos \theta_1 - \mu \cos \theta_2 = 0 \]
\[\lambda \sin \theta_1 - \mu \sin \theta_2 = -s \]

\[\lambda \cos \theta_1 \sin \theta_1 - \mu \cos \theta_2 \sin \theta_1 = 0 \]
\[\lambda \sin \theta_1 \cos \theta_1 - \mu \sin \theta_2 \cos \theta_1 = -s \cos \theta_1 \]
\[-\mu (\cos \theta_2 \sin \theta_1 - \sin \theta_2 \cos \theta_1) = s \cos \theta_1 \]

\[\mu = \frac{s \cos \theta_1}{\cos \theta_2 \sin \theta_1 - \sin \theta_2 \cos \theta_1} \]
\[= \frac{s \cos \theta_1}{\cos (\theta_1 + \theta_2)} \]

References